Mathematical Representation of Articular Surfaces Using Influence Surface Theory

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Many mathematical techniques have been developed to determine the geometry of articular joint surfaces, because of its so importance to the study of human joint biomechanics. However, a three-dimensional geometric model of the articular joint, which is essential to solid modelling, contact area measurement, and load bearing analyses, has not been well developed. This study proposes to define the articular geometry of the distal femoral joint of the human knee. A mathematical method based on the influence surface theory of plates is established to generate representations of three-dimensional articular surfaces. A mathematical cone and the surface of the human distal femur are accurately recreated, allowing their geometric properties to be determined. Results suggest that this method can be an effective tool for representing articular surfaces.

Key Words : Articular Joint Surface Geometry, Influence Surface Theory, Distal Femur, Joint Biomechanics.

1. Introduction

Accurate knowledge of the three-dimensional articular surface geometry in human joints is crucial to the study of joint biomechanics. It affords an understanding of relationships between the articular surface and load bearing, allows development of diagnostic and reconstructive procedures for traumatized or otherwise abnormal joints, and enhances design of artificial joint prostheses. Hence, much interest has focused in recent years on the geometry of articular surfaces (Blankvoort and Huiskes, 1986; Blankevoort et al., 1991; Erkman and Walker, 1974; Garg and Walker, 1990; Huiskes et al., 1985; Kurosawa et al., 1985; Kurosawa et al., 1985; Llewllyn et al., 1989; PV-WAVE Technical Reference Manual, 1990; Scherer and Hillberry, 1979; Shiba et al., 1988).

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Many measurement techniques, such as close-range stereophotogrammetry (Huiskes et al., 1985), and photographs of sliced specimens (Garg and Walker, 1990), have been used to describe articular surfaces. These techniques provide three-dimensional coordinates relative to a reference system of a number of points on an articular surface. The coordinates of these points are not completely continuous, because the continuous articular surface is only approximated by finite points. Accurate representation of the original surface is essential to discover if articular surfaces retain their geometric properties and functions in articular motion. Articular surfaces, however, are usually highly irregular, making it more difficult to describe them mathematically than it is to describe classic geometric surfaces such as cylinders or spheres.

Studies dealing with the representation of articular surfaces (most of which are concerned with human knee joints) describe simple geometric curves (Rehder, 1983), a sphere (Blankvoort and Huiskes, 1986), or a polynomial (Blankevoort et al., 1991) with different degrees. These representations give only a profile or a crude approximation of an actual articular surface; they fail to reflect true three-dimensional geometric properties. The polynomial fit of an articular surface necessitates determination of its term composition and degree, which are heavily dependent on prior understanding of the surface. It is nearly impossible to obtain a satisfactory polynomial with high fitting accuracy. A more sophisticated fitting technique is Coon's bicubic spline (Huiskes et al., 1985; Scherrer and Hillberry, 1979), which is to some extent a flexible mapping method. Because Coon's method is, in essence, piece-by-piece mapping, small regular meshes must be divided on the articular surface to acquire sufficient coordinate data for approximation of the original surface. A need exists for a universal fitting technique to represent appropriately, accurately and effectively, the diversity of articular surfaces. A new method for modelling articular surfaces based on the influence deflection of an infinite plate subjected to lateral loads is described in this paper. This method's significant advantage is its general applicability and flexibility. It is capable of representing any articular surface or other complicated surface shape by using uniform mathematical manipulation.

2. Materials and Methods

2.1 Mathematical Model

Deflection equation of the plate.

Consider a plate that is subjected to lateral forces only. The plate deflection obeys the governing differential equation

$$D\nabla^4 w = q \tag{1}$$

where D represents the bending rigidity of the plate, w is the lateral deflection, and q is the uniform load. Equation (1) is a nonhomogeneous biharmonic equation. Its homogeneous form is

$$\nabla^4 w = 0 \tag{2}$$

Equation (2) corresponds with the presence of lateral concentrated loads. We know that the most general form of the rigorous solution of the nonhomogeneous Eq. (1) can be written as

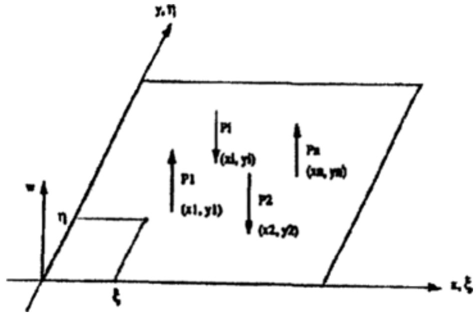


Fig. 1 An infinite plate subjected to a group of concentrated loads. Point (x_i, y_i) is the load point, and point (x, h) is the observed point, in which the deflection is examined.

$$w(\xi,\eta) = w_P(\xi,\eta) + w_H(\xi,\eta) \tag{3}$$

where w_P is a particular solution of the nonhomogeneous Eq. (1), and wH is the solution of the homogeneous Eq. (2), with both defined in the $\xi\eta$ plane.

Influence deflection of the infinite plate under concentrated loads.

Consider an infinite plate of arbitrary shape (Fig. 1) Assume that concentrated loads P_1, P_2, $\cdots$, P_N are acting at points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n). Then according to Maxwell's reciprocity law and the singular solution of plate Eq. (2) (Szilard, 1974), the deflection surface generated by P_i is :

$$w_i(\xi, \eta) = kP_i[(\xi - x_i)^2 + (\eta - x_i)^2] \cdot \mathrm{Ln}[(\xi - x_i)^2 + (\eta - x_i)^2] \tag{4}$$

where point (ξ, η) is an arbitrary observation point on the plate, k is the bending rigidity which is assumed to be a proportional coefficient ($k=1$) in this study, and point (x_i, y_i) represents the load point of the ith individual load Pi.

The influence deflection for all loads, which represents a particular solution of the plate equation, is the sum of the function wi(ξ, η) for each of the loads:

$$w_P(\xi, \eta) = \sum w_i(\xi, \eta) \tag{5}$$

Suppose that the deflection of the infinite plate tends to be flat at distances far away from the loads. A linear plane function is then chosen as the homeogenous solution of plate Eq. (2):

$$w_H(\xi, \eta) = A + B\xi + C\eta \tag{6}$$

where A, B, and C are all coefficients.

The entire solution of the plate under the concentrated load group is the combination of Eq. (5) and (6) according to Eq. (3):

$$w(\xi, \eta)=A+B\xi+C\eta+\sum w_i(\xi, \eta) \quad (7)$$

Evidently, if Eq. (7) is weth posed, the loads acting on the infinite plate must satisfy the equilibrium conditions. Because they constitute a system of spatial parallel forces, the following three equations can be used:

$$\sum P_i=0 \quad (8)$$

$$\sum P_i x_i=0 \quad (9)$$

$$\sum P_i y_i=0 \quad (10)$$

Determination of surface geometry.

In a description of articular surfaces, three-dimensional coordinates of sampling points on an articular surface can be obtained by using different measuring techniques. Assume that $w_s(x_i, y_i)$ $(i=1, 2, \cdots, N)$ are the sampled height coordinates of the articular surface. If the height coordinates of the articular surface are considered deflections of the infinite plate defined by Eq. (7), then analogously we get N linear equations with the unknown A, B, C, and P_i $(i=1, 2, \cdots, N)$:

$$A+Bx_i+cy_i+\sum w_i(x_i, y_i)=w_s(x_i, y_i) \quad (11)$$

Combined with the three additional Eq. (8), (9), and (10), a total of N+3 equations are formed. Their unknown coefficients can be solved by utilizing a numerical computation technique. Once the coefficients are determined, we then generate a representation of the articular surface as defined by Eq. (7).

Calculation of volume and surface area.

Because the articular surface is now expressed by Eq. (7), the volume of the surface, defined as that circumscribed by the surface and the coordinate plane, can be written as (Wylie and Barrett, 1982):

$$V_t=\int_\Omega\int w(\xi, \eta)\,d\xi d\eta \quad (12)$$

or more explicitly as:

$$V_t=\int_\Omega\int\{A+B\xi+C\eta+\sum w_i(\xi, \eta)\}\,d\xi d\eta \quad (13)$$

Here Ω is the integral area on the coordinate plane $\xi\eta$. Also, the expression for the surface area is readily given by (Wylie and Barrett, 1982):

$$S_t=\int_\sigma\int\sqrt{1+\left(\frac{\partial w}{\partial\xi}\right)^2+\left(\frac{\partial w}{\partial\eta}\right)^2}\,d\xi d\eta \quad (14)$$

where σ is the projected area of the surface on the coordinate plane $\xi\eta$.

Expanding the two partial derivatives, we have:

$$S_t=\int_\sigma\int\sqrt{1+E^2+F^2}\,d\xi d\eta \quad (15)$$

where

$$E=\{B+2\sum_{i=1}^{N}P_i(1+E_n[(\xi-x_i)^2+(\eta-y_i)^2])(\xi-x_i)\}^2$$

$$F=\{C+2\sum_{i=1}^{N}P_i(1+\mathrm{Ln}[(\xi-x_i)^2+(\eta-y_i)^2])(\xi-x_i)\}^2$$

Equations (12) and (13) are the general equations for calculating the volume and area of articular surfaces. However, if the above integral areas are irregular, as in most articular surfaces, the integrals cannot be calculated directory, because the integral areas are inexpressible analytically. Even if the integral areas are regular, the two analytical integrals (13) and (15), especially the latter, are quite complex. Hence, numerical forms of the algorithms of the two integrals are necessary.

Let the integral areas in (13) and (15) be divided into A1, A2, . . . Ak small regions. Each of the regions is then divided into MN small elements (Fig. 2) . The height coordinate of any small element is expressed by an arithmetic average of its four node-point height coordinates, so that

$$V_n=\sum_{i=1}^{K}\sum_{j=1}^{M\times N} w_A(\xi_{ij}, \eta_{ij})\Delta\sigma_{ij} \quad (16)$$

where V_n denotes the volume as calculated by the numerical method and $w_A(\xi_{ij},\eta_{ij})$ and $\Delta\sigma_{ij}$ denote the averaged value of the jth height coordinate defined by Eq. (7) and the jth element area for the jth region, respectively.

To calculated the surface area, we also divide the projected integral area into K small regions,

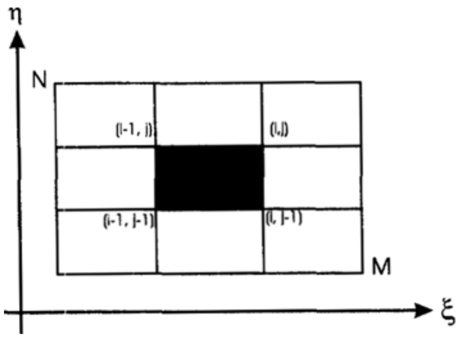


Fig. 2 In the rectangular coordinate system, an integral area is divided into M×N small elements, and the height coordinate $w(\zeta, \eta)$ of each of the elements is represented by an average of the four node-point coordinates.

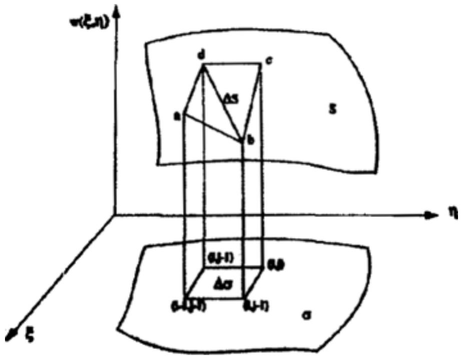


Fig. 3 Illustration for determining the surface area. S denotes the surface, σ denotes its projected area, $\varDelta\sigma$ represents any divided element of the area, $\varDelta S$ represents the corresponding surface element divided into two triangles abd and bcd.

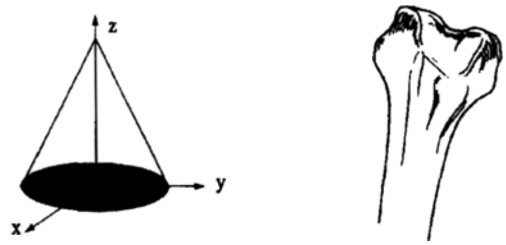
with each of the regions further divided into M×N small elements $\varDelta\sigma$. The $\varDelta\sigma$ corresponds to a spatial quadrilateral, which is divided once again into two triangular elements (Fig. 3). The surface area is thus approximated by the sum of the triangular elements; that is,

$$S_n=\sum_{i=1}^{K}\sum_{j=1}^{M\times N}\varDelta S_{ij}=\sum_{i=1}^{K}\sum_{j=1}^{M\times N}\varDelta S(_{abd})_{ij}+\varDelta S(_{bcd})_{ij} \tag{17}$$

where the triangular areas are defined by

$$\varDelta S_{(d_1 d_2 d_3)ij}=\sqrt{L(L-d_1)(L-d_2)(L-d_3)} \tag{18}$$

where $L=(d_1+d_2+d_3)/2$, and the $d_m(m=1, 2, 3)$ are lengths, readily determined by two-point distances calculated from their coordinates, espe-



(a) (b)

Fig. 4 Illustration of mathematical cone model (a) and regular geometric articular surface (b).

cially for a large number of sampling points. By using the PV-WAVE software package (Precision-Visuals, Technical Reference Manual, 1990), we designed a calculation program capable of reconstructing articular surfaces with graphical representation on a computer screen in different view angles, and that can evaluac the surface area as well as the volume enclosed by the surface and some specified coordinate plane.

3. Task Tests

3.1 Task A

A mathematical cone (Fig. 4(a)) can be represented with the equation:

$$Z(X, Y)=1-\sqrt{X^2+Y^2} \tag{19}$$

In the XY plane, Eq. (19) correspords to a circular area with radius 1. To obtain the mathematical representation of the cone surface, we chose 25 and then 81 points by regularly dividing the circular area into a meshed pattern. The X and Y coordinates, of mesh node points were readily determined, and their Z coordinates were then decided by Eq. (19). With three-dimensional coordinates obtained from Eq. (19), surface fitting was performed using our new method, based on the influence surface theory of plates to obtain a representation of the cone surface.

3.2 Task B

Task A created a representation of a regular geometric surface (Fig. 4(b)) in which no data errors were implicated. Typical articular surfaces, however, are highly irregular, and require sampling of surface coordinates using different measurement techniques. In view of this, a more practical problem, determination of a distal femoral surface, which is believed to be one of the complicated human joint surfaces, was investigated. A commercially available plastic femur model was mounted on the frame of a custom-designed three-dimensional mechanical digitizer (Colbaugh et al., 1991). By moving the stylus of the digitizer on the distal femoral surface, three-dimensional coordinates of the surface points with respect to a fixed global coordinate system of the digitizer, were acquired. The digitizing track, or moving path of the stylus on the surface, was arbitrary. A total of 396 sampling points were digitized on the femoral surface (Fig. 5). All these procedures were completed in a SUN workstation, which had been installed with the calculation program based on PV-Wave package software.

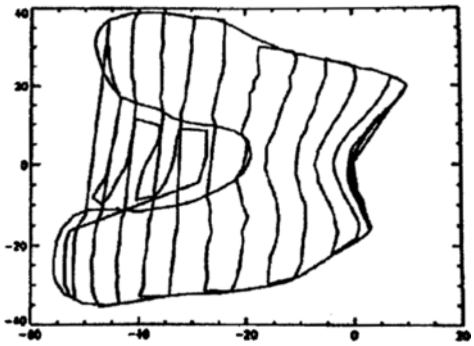
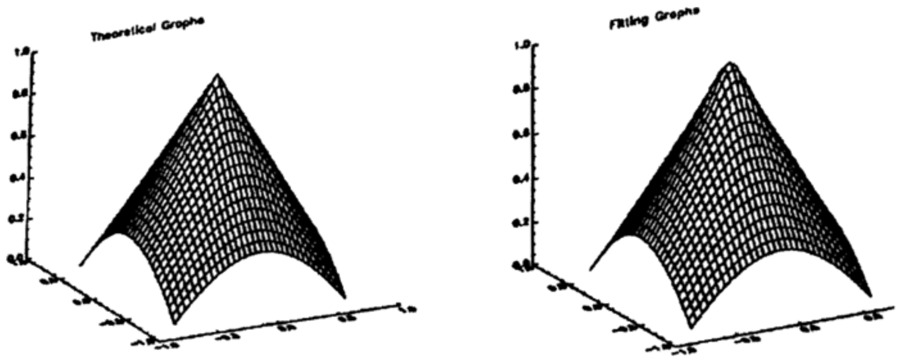


Fig. 5 The digitizing track on the distal femoral surface.

4. Results

To verify the effectiveness as well as applicability of this new technique, two examples, a mathematical cone and a femur model, are presented. Figure 4(a) gives a theoretical graph, defined by Eq. (19), and a fitting graph, represented by Eq. (7), for the 81 coordinates based on predicted curve fitting to prove its effectiveness. The theoretical and calculated volume and surface area of the cone model are listed in Table 1. Comparison of the two mathematical cone graphs in Fig. 6(a) and 6(b) indicate good conformity between the representational and theoretical cones. In addition, Table 1 shows that the calculated cone

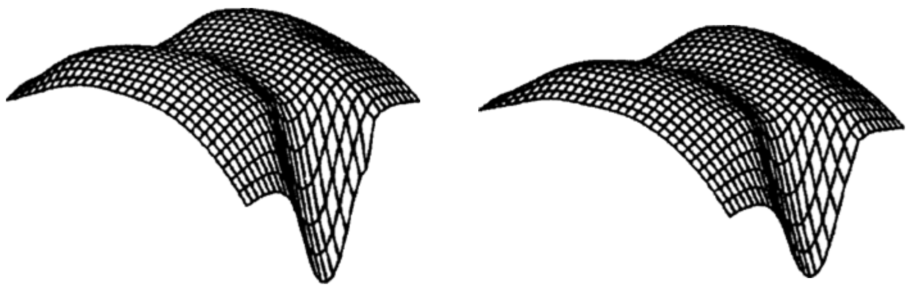


(a) (b)

Fig. 6 The theoretical (a) and representational surface (b) of a mathematical cone. A total of 81 sampling points were used in (b).

Table 1 Comparison of theoretical and representational volume and surface of mathematical cone model with different sampling points.

	Theoretical model	25 Sampling points		81 Sampling points	
		Calculated	Error %	Calculated	Error %
Volume	1.05	1.01	3.81	1.02	2.86
Surface area	4.44	4.30	3.15	4.41	0.64



(a) (b)

Fig. 7 Representations of the distal femoral surface with the proposed method. (a) 396 sampling points, (b) 198 sampling points.

Table 2 Surface area of the distal femoral surface with different sampling points.

Unit (Cm^2)	198 Sampling points	396 Sampling points
Surface area	31.7	32.4
Projected area	27.4	27.7
Ratio	1.16	1.17

volume and surface areas were close to the theoretical points from 25 to 81, the calculated cone volume and surface areas were even closer to the theoretical volume and area.

Figures 7(a) and 7(b) provide a graphic representation of the femoral surface. Similar graphic representations were observed under different conditions. It should be noted that Figure 5(a) was defined by the use of 396 sampling points, whereas Figure 5(b) was created by selecting half of the 396 points alternatively to examine quantitative differences between both conditions. These alternate 198 points out of 398 points were obtained by a numerical interpolation method. Table 2 gives femoral surface areas, including the projected are a, defined as the projection of the femoral surface onto the horizontal plane, along with the areas for both 396 and 198 sampling points. The projected area was approximated by a 12-sided polygon, which allowed the femoral surface area to be computed by dividing the polygon into 400 small rectangles. Each was further divided into 225 smaller rectangular elements. For two different sampling points, the computed surface areas are quite similar to each other quantitatively. The average ratio between the computed surface area and the projected area was 1.17, indicating that the actual femoral surface area was approximately 17% greater than its projected area.

5. Discussion

We have described a new technique for representing three-dimensional articular surfaces using the influence surface theory of plate. Our method has some advantages that can overcome various defects of other mathematical fitting techniques (Blankevoort et al., 1991; Blankevoort and Huiskes, 1986; Huiskes et al., 1985; Rehder, 1983; Scherrer and Hillverry, 1979). The technique developed in this study combines a linear plane function, a series of modified natural logarithm functions, and the differential equation for deflection of a plate under concentrated loads. Because the adopted functions satisfy the deflection differential equation (a double harmonic equation), they are all biharmonic functions, capable of generating approximations of the original surface. Ruan (1987) used these same biharmonic functions as trial functions for solving collocation problems in surveying (Ruan, 1987).

The method developed in this study has several advantages for representing articular surfaces. First, the expression for representing articular surfaces as shown in Eq. (7) is independent of the selection of the origin of the coordinate system. This is superior to the polynomial fitting technique in which the expression is different for different origins and dependent on the degree of the polynomial chosen. Second, no specific data sampling pattern is required. Furthermore, 70% fewer selected sampling points can still lead to a good representation of the articular surface based on the quantity of volume and surface area (Table 1). comparing the of theoretical quantity of volume and surface area, 25 sampling points least to a difference of 3.81% in volume and 3.15% in surface area, while 81 sampling points leads to 2.86% and 0.68%, respectively. Fitting accuracy is usually increased with an increase in the number of sampling points. Finally, an integral mathematical expression can be automatically established when obtaining the mathematical representation of the joint surface, which is more convenient than other fitting techniques in determining articular surface geometry as well as in assessing articular joint study.

This study details two examples of the application of our method. In task A, comparing Figure 4(a) with Figure 4(b), our method fits the entire mathematical cone except for its top points, where the representational graph is slightly different from the original. This occurred because the original sharp point on the top has a singularity mathematically that is non-differentiable due to the concentrated loading as boundary condition. However, it has been proven in this study that if the number of sampling points around the apex is increased, a higher fitting accuracy can be obtained, which would be almost impossible with other fitting techniques (Blankevoort et al., 1991; Blankevoort and Huiskes, 1986; Huiskes, et al., 1985; Rehder, 1983; Scherrer and Hillberry, 1979)

In task B, we reconstructed the geometry of a representation of the femoral surfaces. The mathematical representation of articular surface was described with biharmonic functions in our study, even though most of previous studies had adopted the two spherical function to describe the articular surface of the femur (Blankevoor and Huiskes, 1986). We believe that mathematical representations using biharmonic function may be more versatile in describing the geometry properties of articular joints than other methods. It may also more conveniently provide the geometric properties of surface area and volume, along two orthogonal directions, because the integral mathematical expression of the femoral surface is already defined. As a by-product, the ratio between the computed femoral surface area and its projected area was provided in task B.

This may make it possible, should enough femurs be studied, to describe a better statistical relationship between the two areas. The statistically proven ratio could be used for estimation of the femoral area according to its projected area.

Two different sets of sampling points were used in reconstructing the femoral surface to discover whether fewer sampling points would decrease the quality of femoral surface reconstruction. Statistical analysis shows that the average value of the absolute difference between interpolated and measured coordinates with 198 sampling points case is 0.68 ± 1.27 mm. These two sets generated representations that predict surface area of the articular joint and lead to a difference of 2% in the surface area. However, this interpolation may be accepted if we consider the measurement accuracy (0.5 mm with resolution of 0.2 mm) of our custom-designed three-dimensional digitizer. This suggests that the method may be an effective two-dimensional interpolation, which can lead to good representation of articular surface with a smaller number of sampling points. This is of great importance because it can decrease the number of specimen sections required to apply the technique of the photographs of sliced specimens (PSS), where numerous sections are required to reconstruct three-dimensional surface geometry.

Theoretical and numerical expressions calculating surface area are given in Eqs. (15) and (17). We found that the numerical expression gave more stable and accurate results than did the theoretical expression when different divisions were chosen in the calculated area. Theoretical expression problems probably result from first-order partial derivatives, which are highly sensitive to data noise that inevitably exists in sampling data. Hence, use of the theoretical forms could lead to a relatively large calculation error. It is evident, therefore, that numerical expression

(17) is preferable to the theoretical expression when evaluating articular surface area.

It should be noted that in this mathematical method the representational surface must pass through all sampling points, the coordinates and measurement accuracies of which are acquired and controlled by the three-dimensional digitizer measurement technique. Therefore, to obtain more accurate representation of the articular surface with this method, a higher accuracy measurement must be implemented. In determining distal femoral surface, we used the three-dimensional digitizer, the measurement accuracy of which was approximately 0.5 mm. This level of accuracy is acceptable, because the example is only meant to show the applicability of this method. In addition the size of the femur was relatively large. Should small articular surfaces, for example, carpal bones, be studied using this method, a more discriminating measurement technique is deemed necessary for accurate representation.

One should be aware that since the technique includes an $N+3$ linear equation set whose coefficients must be solved for the surface representation, and that the number of equations is directly proportional to that of the sampling points. Thus, a vast coefficient matrix of the equation set will be formed if a large number of sampling points is acquired and used. This matrix would occupy a great amount of computer memory, consume a great deal of computing time, and possibly make it impossible to perform the computation. Therefore, when applying this method and using numerous sampling points, a powerful computer is required.

6. Conclusion

A new technique for representing articular surfaces based on the influence surface theory of plates is introduced here. Two examples, a mathematical cone and femur model, were used to verify technique effectiveness and applicability. Results showed that the proposed method is a general and flexible fitting technique well-suited for accurate representation of any articular surface or other complicated surface shape. With few limitations, it can be an effective tool in determining articular surface geometry.

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